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Algebraic properties and symmetries of the symmetric Ashkin-Teller model

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Abstract. We investigate the automorphic properties of the partition function of an anisotropic but symmetric Ashkin-Teller model on a square lattice. We discuss the structure and the location of critical and solubility lines of the isotropic model in the present framework and comment on their algebraic and/or non-algebraic character.

1. Introduction

Much attention is currently being devoted to the study of the phase diagram of multi-state spin models in two dimensions, perhaps due to the belief, according to the Migdal-Kadanoff renormalisation scheme, that lattice gauge theories should have the same phase structure (Alcaraz and Köberle 1981). However it has been shown that this connection is not rigorous (Creutz and Okawa 1983). We are interested in studying, in this paper, multi-state spin systems for their own sake since they do contain in themselves a host of information. Their critical, or nearly critical, properties have been treated in the past by the renormalisation group (Kohmoto *et al* 1981), by variational methods (Rujan *et al* 1981) and recently with the help of the conformal group, as advocated by Belavin *et al* (1984). The simplest model at hand which depends on two parameters is the symmetric Ashkin-Teller model (SAT). Its phase diagram is rather complex and has been worked out numerically by series analysis (Ditzian *et al* 1980) by Monte Carlo simulations (Baltar *et al* 1984) and by the finite-size scaling method (Igloi and Solyom 1984) in addition to some previous numerical work (Ashley 1978).

In the following we shall reconsider this model theoretically. Although it was introduced some forty years ago (Ashkin and Teller 1943) it still remains a fascinating topic for research because of its rich critical properties presenting multi-criticality and bifurcation of the phases. The problem of locating exactly its bifurcating critical lines is still unresolved. Here, we seek to probe into its structure by using the inverse relation for the partition function in conjunction with all its existing symmetries. Such a program has been carried out for some vertex models (Stroganov 1979, Schultz 1981)

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and for some spin models with 'interaction round a face' (IRF models) (Baxter 1982a, b) as well as for the anisotropic q-state Potts model (Jaekel and Maillard 1982).

2. The symmetrical Ashkin–Teller model

A convenient way of describing the SAT model (Fan 1972a) is to consider two sets of Ising spins per site σ_i and τ_i such that the interaction between two neighbouring sites is given by the statistical weight:

$$w_{ij}^{\alpha} = \exp[K_\alpha(\sigma_i\sigma_j + \tau_i\tau_j) + K''_\alpha \sigma_i\sigma_j\tau_i\tau_j - 2K_\alpha - K''_\alpha]$$

where $K_\alpha = E_\alpha/kT$ and $K''_\alpha = E''_\alpha/kT$, $1/kT$ being the inverse temperature, E_α and E''_α are the two- and four-body energies in the directions $\alpha = \mathrm{v, h}$ (vertical/horizontal) of the square lattice. It turns out that the natural quantities arising in the discussion are:

$$\omega_\alpha = \exp(-2K_\alpha - 2K''_\alpha)$$

$$\omega''_\alpha = \exp(-4K_\alpha).$$

Let $Z(\omega_\mathrm{v}, \omega''_\mathrm{v}|\omega_\mathrm{h}, \omega''_\mathrm{h})$ be the partition function per site. It remains clearly invariant under a 90° rotation of the lattice:

$$Z(\omega_\mathrm{v}, \omega''_\mathrm{v}|\omega_\mathrm{h}, \omega''_\mathrm{h}) = Z(\omega_\mathrm{h}, \omega''_\mathrm{h}|\omega_\mathrm{v}, \omega''_\mathrm{v})$$

which amounts to interchange $(\omega_\mathrm{v}, \omega''_\mathrm{v}) \leftrightarrow (\omega_\mathrm{h}, \omega''_\mathrm{h})$, hereafter called S.

Since one can flip the sign of the coupling constants K_α and the spins σ_i and τ_i on every other lattice site without altering w_{ij}^α we see that $Z(\omega_\mathrm{v}, \omega''_\mathrm{v}|\omega_\mathrm{h}, \omega''_\mathrm{h})$ remains invariant under a transformation T defined as

$$T: (\omega_\alpha, \omega''_\alpha) \to (\omega_\alpha^T = \omega_\alpha/\omega''_\alpha, \omega''^T_\alpha = 1/\omega''_\alpha).$$

The SAT model also satisfies a duality symmetry (Fan 1972b) given by

$$D: (\omega_\alpha, \omega''_\alpha) \to (\omega^*_\alpha, \omega''^*_\alpha)$$

where $\alpha \neq \alpha' = (\mathrm{v, h})$

$$\omega^*_\alpha = \frac{1 - \omega''_{\alpha'}}{1 + \omega''_{\alpha'} + 2\omega_{\alpha'}}$$

$$\omega''^*_\alpha = \frac{1 + \omega''_{\alpha'} - 2\omega_{\alpha'}}{1 + \omega''_{\alpha'} + 2\omega_{\alpha'}}$$

and

$$Z(\omega_\mathrm{v}, \omega''_\mathrm{v}|\omega_\mathrm{h}, \omega''_\mathrm{h}) = Z(\omega^*_\mathrm{v}, \omega''^*_\mathrm{v}|\omega^*_\mathrm{h}, \omega''^*_\mathrm{h}).$$

There is also another symmetry N corresponding to negating ω_α (or $K''_\alpha \to K''_\alpha$ and $K_\alpha \to K_\alpha + \frac{1}{2}\mathrm{i}\pi$ under suitable boundary conditions), i.e. Z is an even function of ω_α.

Finally arguing as in the standard anisotropic q-state Potts model (Jaekel and Maillard 1982) we find that the SAT partition function obeys the functional equation:

$$Z(\omega_\mathrm{v}, \omega''_\mathrm{v}|\omega_\mathrm{h}, \omega''_\mathrm{h}) Z(\underline{\omega}_\mathrm{v}, \underline{\omega}''_\mathrm{v}|\underline{\omega}_\mathrm{h}, \underline{\omega}''_\mathrm{h}) = (1 + 2\omega_\mathrm{v}\underline{\omega}_\mathrm{v} + \omega''_\mathrm{v}\underline{\omega}''_\mathrm{v})$$

where the inverse weights $\underline{\omega}_\alpha$ and $\underline{\omega}''_\alpha$ are obtained from the inverse transform I

$$I: (\omega_\alpha, \omega''_\alpha) \to (\underline{\omega}_\alpha, \underline{\omega}''_\alpha)$$

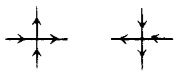
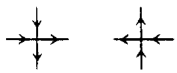
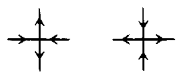
where

$$\underline{\omega}_{\rm h} = \omega_{\rm h}^{-1} \qquad \underline{\omega}''_{\rm h} = (\omega''_{\rm h})^{-1}$$

$$\underline{\omega}_{\rm v} = \omega_{\rm v}\frac{1-\omega''_{\rm v}}{2\omega_{\rm v}^2-(1+\omega''_{\rm v})} \qquad \underline{\omega}''_{\rm v} = \omega''_{\rm v}\frac{1+\omega''_{\rm v}-2\omega_{\rm v}^2/\omega''_{\rm v}}{2\omega_{\rm v}^2-(1+\omega''_{\rm v})}.$$

The present parametrisation (by ω_α and ω''_α) of our transformations *S*, *T*, *D*, *I*, *N*, which are involutions, is cumbersome due to the presence of rational functions. Recalling that the SAT model is also equivalent to a staggered six-vertex model (Kohmoto *et al* 1981, Wu 1977) whose weights are given in table 1.

Table 1. Weights of the staggered six-vertex system equivalent to the SAT model.

Vertex configuration $\alpha = {\rm v, h}$	 a_α	 b_α	 c_α
Sublattice (h)	$\sinh 2K_{\rm h}$	$\exp(-2K''_{\rm h})$	$\cosh 2K_{\rm h}$
Sublattice (v)	$\exp(-2K''_{\rm v})$	$\sinh 2K_{\rm v}$	$\cosh 2K_{\rm v}$

We can go over the variables Δ_α and x_α defined by

$$\Delta_\alpha = \frac{a_\alpha^2+b_\alpha^2-c_\alpha^2}{2a_\alpha b_\alpha} = \frac{\omega_\alpha - \omega''_\alpha/\omega_\alpha}{1-\omega''_\alpha}$$

$$x_\alpha = a_\alpha/b_\alpha \qquad \text{i.e. } x_{\rm h} = \frac{1-\omega''_{\rm h}}{2\omega_{\rm h}} \qquad \text{and} \qquad x_{\rm v} = \frac{2\omega_{\rm v}}{1-\omega''_{\rm v}}.$$

Δ_α is in fact the Lieb's invariant of a six-vertex model and remarkably its functional form remains invariant under *S*, *D*, *I* (up to a sign for *T*). We thus express our transformations as

$$S:\begin{pmatrix}\Delta_{\rm v}, x_{\rm v}\\ \Delta_{\rm h}, x_{\rm h}\end{pmatrix} \to \begin{pmatrix}\Delta_{\rm h}, x_{\rm h}\\ \Delta_{\rm v}, x_{\rm v}\end{pmatrix}$$

$$D:\begin{pmatrix}\Delta_{\rm v}, x_{\rm v}\\ \Delta_{\rm h}, x_{\rm h}\end{pmatrix} \to \begin{pmatrix}\Delta_{\rm v}^* = \Delta_{\rm h}, x_{\rm v}^* = x_{\rm h}^{-1}\\ \Delta_{\rm h}^* = \Delta_{\rm v}, x_{\rm h}^* = x_{\rm v}^{-1}\end{pmatrix}$$

$$I:\begin{pmatrix}\Delta_{\rm v}, x_{\rm v}\\ \Delta_{\rm h}, x_{\rm h}\end{pmatrix} \to \begin{pmatrix}\Delta_{\rm v}, 2\Delta_{\rm v}-x_{\rm v}\\ \Delta_{\rm h}, (2\Delta_{\rm h}-x_{\rm h})^{-1}\end{pmatrix}$$

$$N, T:\begin{pmatrix}\Delta_{\rm v}, x_{\rm v}\\ \Delta_{\rm h}, x_{\rm h}\end{pmatrix} \to \begin{pmatrix}-\Delta_{\rm v}, -x_{\rm v}\\ -\Delta_{\rm h}, -x_{\rm h}\end{pmatrix}$$

S and *I* generate an infinite discrete group G, isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}$ with elements of the form $I^\mu(SI)^\nu$, $\mu = 0, 1$, and $\nu \in \mathbb{Z}$. Observe that *T* and *D* commute with G.

According to the general concepts put forward in Jaekel and Maillard (1982) a set of distinguished points (such as critical points, soluble points, zeros of partition function etc, . . .) is necessarily transformed into themselves under the action of all elements (or an infinite subgroup) of G and consequently also by the combined action of G with

T and D. A proposed, presumed or suspected critical or integrable manifold in the phase diagram can only make sense if it remains globally invariant under G ultimately extended by T and D. We seek thus to construct such manifolds on which the partition function may presumably be evaluated exactly (for instance, as in many well known cases, in terms of simple infinite Eulerian products).

To this end we introduce the following new variables adapted to the automorphy group G

$$y_\alpha = \frac{x_\alpha - q_{+,\alpha}}{x_\alpha - q_{-,\alpha}} \qquad \text{with } q_{\pm,\alpha} = \Delta_\alpha \pm (\Delta_\alpha^2 - 1)^{1/2}$$

$q_{\pm,\alpha}$ are unimodular for $|\Delta_\alpha| < 1$ and real for $|\Delta_\alpha| > 1$. Then I and S are now expressed as:

$$I: \begin{pmatrix} \Delta_{\mathrm{v}}, y_{\mathrm{v}} \\ \Delta_{\mathrm{h}}, y_{\mathrm{h}} \end{pmatrix} \to \begin{pmatrix} \Delta_{\mathrm{v}}, y_{\mathrm{v}}^{-1} \\ \Delta_{\mathrm{h}}, q_{+\mathrm{h}}^4 y_{\mathrm{h}}^{-1} \end{pmatrix}$$

$$(SI)^2: \begin{pmatrix} \Delta_{\mathrm{v}}, y_{\mathrm{v}} \\ \Delta_{\mathrm{h}}, y_{\mathrm{h}} \end{pmatrix} \to \begin{pmatrix} \Delta_{\mathrm{v}}, q_{+\mathrm{v}}^4 y_{\mathrm{v}} \\ \Delta_{\mathrm{h}}, q_{+\mathrm{h}}^{-4} y_{\mathrm{h}} \end{pmatrix}.$$

The last transformation generates an infinite distinguished discrete subgroup H. Let us consider the algebraic varieties globally invariant under H. Using an argument of Jaekel and Maillard (1982) we see that they are necessarily of the form: $y_{\mathrm{h}}^n y_{\mathrm{v}}^m = \mathbb{C} =$ constant and $q_{+\mathrm{h}}^{4n} = q_{+\mathrm{v}}^{4m} = q_+^{4n}$ (n, m integers). If such a curve has to remain globally invariant under S *and* I separately one should require that $n = m$ and $q_{+\mathrm{h}} = q_{+\mathrm{v}}$ (or equivalently $\Delta_{\mathrm{h}} = \Delta_{\mathrm{v}} = \Delta$). It follows that:

$$y_{\mathrm{h}} y_{\mathrm{v}} = \pm q_+^4.$$

The restriction of this equation to the isotropic SAT model yields three curves in the original parameters ω, ω'')

$$\omega'' + 2\omega = 1 \tag{1}$$

$$\omega'' - 2\omega = 1 \tag{2}$$

$$[(1-\omega'')^2 - 4\omega^2]^2 = (1+\omega'')^2[(1-\omega'')^2 + 4\omega^2]. \tag{3}$$

Equation (1) represents the already known self-dual line whereas equation (2) is the T transform of the self-dual line. The fourth-order algebraic curve lies in the 'unphysical' area of the phase diagram as shown in figure 1. It intersects the $\omega'' = 0$ axis at $\omega = \sqrt{3}/2$. For $\omega'' > 0$ it is located in the region where the dual weights are negative, it goes through the high-temperature point $\omega'' = \omega = 1$ and is tangential there to the 'duality envelope' $1 + \omega = 2\omega''$. The curve continues in the region $\omega'' < 0$ and shows a bifurcation at $\omega = 1$, $\omega'' = -1$ where two branches meet tangentially to the self-dual line and are mapped into each other by duality. Remarkably this curve has been obtained before (Truong 1984) as an integrability curve of a staggered six-vertex model by one Bethe ansatz, where the fifth and sixth vertices carry electric field equal to the spectral parameter of the Baxter parametrisation. Along such a curve the partition function is simply the product of the partition functions corresponding to the two types of six-vertex systems. However, then the role of the electric field on the fifth and sixth vertices becomes absolutely irrelevant, this is the reason why the result holds for the vertex weights of table 1 of the SAT model. The set of these three curves may

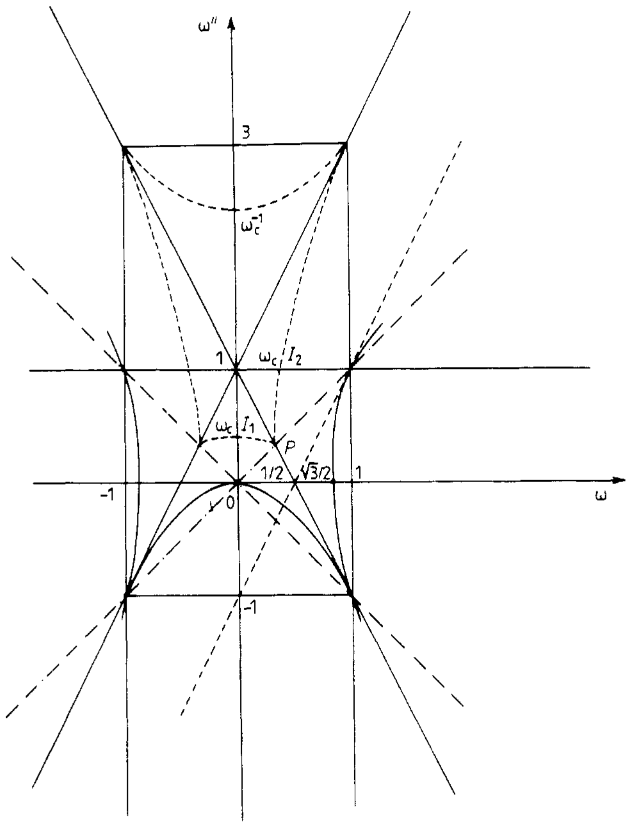


Figure 1. Integrability and criticality lines of the symmetric Ashkin–Teller model.

be also recovered very simply by making use of the notion of 'extended dualities' (Truong 1985) with the assumption $\Delta_v = \Delta_h = \Delta$.

To have an idea of what happens in the anisotropic model, let us consider an 'extreme anisotropic' limit (Kohmoto *et al* 1981). There the parametrisation is as follows:

$$K_h = \tau\beta \qquad \omega_h = \exp[-2\tau\beta(1+\lambda)] \qquad \Delta_h = -\frac{\sinh 2\tau\beta\lambda}{\sinh 2\tau\beta}$$

or or

$$K''_h = \tau\beta\lambda \qquad \omega''_h = \exp(-4\tau\beta) \qquad x_h = \exp(2\tau\beta\lambda)\sinh 2\tau\beta$$

$$K_v = \tfrac{1}{4}\ln(1/\tau\lambda) \qquad \omega_v = \tau \qquad \Delta_v = -\lambda$$

or or

$$K''_v = \tfrac{1}{4}\ln(\lambda/\tau) \qquad \omega''_v = \tau\lambda \qquad x_v = 1/2\tau.$$

The extreme anisotropic limit is reached for $\tau \to 0$. Hence

$$\Delta_h = \Delta_v = \Delta = -\lambda \qquad x_v x_h = \beta \qquad \text{and} \qquad (x_v + x_h) = 2\gamma \to \infty.$$

The equation $y_v y_h = \pm q_+^4$ can be written now as:

$$\Delta(1-\beta)[-\Delta(1+\beta)+2\gamma] = 0.$$

We thus get $\Delta = 0$, the free-fermion condition of Wu and Lin (1975), $\beta = 1$ the self-dual surface and its T transform and the surface

$$\lambda(1+\beta)+2\gamma=0$$

in an extended coordinate system of Kohmoto *et al*: (λ, β, γ).

Let us now look at the case where the automorphy group G degenerates into a finite group. This happens whenever $\Delta = \cos(k\pi/4m)$, k and m being integers. In the case of the isotropic SAT model some of these curves seem to be of special interest. We have just seen that $\Delta = 0$ corresponds to the free-fermion condition, hence soluble by Pfaffian methods. Using the mapping of the q-state Potts model to a staggered six-vertex model (Temperley and Lieb 1971, Baxter *et al* 1976), $\Delta = -\frac{1}{2}$ and $\Delta = -1/\sqrt{2}$ can be seen to correspond to the $q = 1$ and $q = 2$ soluble cases of the Potts model. If $\Delta = -1$ we recover the four-state Potts model, soluble only at criticality. The family of curves $\Delta =$ constant corresponds to a natural foliation of the space of parameters into 'thermodynamic' trajectories joining the low-temperature point ($\omega = \omega'' = 0$) to the high-temperature point ($\omega = \omega'' = 1$) which are invariant under duality.

Our present considerations of the group G do not seem to lead us to the two critical lines bifurcating in the $\Delta < -1$ region from the self-dual point of the four-state Potts model as conjectured by Wegner (1972) and supported by numerical evidence (Ashley 1978, Ditzian *et al* 1980, Baltar *et al* 1984, Igloi and Solyom 1984). It is perhaps useful to review the status of these lines, traced in broken curves on figure 1 as PI_1 and PI_2, their extensions to the whole plane (ω, ω'') being obtained by repeated applications of T and D. The neighbourhood of I_1 has been studied by Kohmoto *et al* (1981) who showed by perturbation that the line starts horizontally and seems to support the belief that it is a line of transition points of Ising type. The neighbourhood of P has been studied by Kadanoff (1981) with the help of the renormalisation group. He shows that the line can be represented by the equations, in the region $\omega'' \geqslant \omega$:

$$(2\omega+\omega''-1)=\pm A\left(\frac{\omega''}{\omega}-1\right)^{1/2}\exp\left[-\frac{3\pi^2}{4\sqrt{2}}\left(\frac{\omega''}{\omega}-1\right)^{-1/2}\right]. \tag{4}$$

If we do not dispute the validity of the renormalisation group, which assumes that the trajectories are analytic near the Gaussian fixed line, we must conclude that these curves cannot be algebraic. Consequently they cannot be associated with integrability curves since the solutions of the Yang–Baxter equations for integrability always lead to algebraic manifolds. At this point, taking into account all the local symmetries near the point P and the general G invariance of the model we may venture to suggest that equation (4) should be replaced by

$$(x^2-1)(x^{-2}-1)=\pm A'(\Delta^2-1)^{1/2}\exp\left(-\frac{4\pi^2}{(\Delta^2-1)^{1/2}}\right). \tag{5}$$

For $\omega \sim \omega'' \sim \frac{1}{3}$ it describes the same behaviour as (4).

To make such a non-analytic behaviour plausible we shall give an illustration extracted from the study of an asymmetric six-vertex model (Lieb and Wu 1972) which consists in arguing in the neighbourhood of integrability. We recall that, as a consequence of the ice rule, polarisation (or fraction of reversed arrows in a row) is a conserved quantity even in the presence of an electric field V. The free energy per site is then expressed as

$$-\beta f = \max_{-1\leq y\leq 1} (Z(y)+yV)$$

where $Z(y)$ is the expression generalising $Z(0)$, the free energy of the zero-field six-vertex model for arbitrary y. The optimum choice of $-\beta f$ is then defined by the condition

$$Z'(y) = -V.$$

To locate a phase transition, whereby y flips from a zero to a non-zero value, we set thus $Z'(0) = -V$. The $Z'(0)$ is obtainable through the Bethe ansatz integral equation as an Eulerian product in terms of the nome of an elliptic function $q = e^{-\lambda}$, where λ is connected to the Lieb invariant by $\Delta = -\cosh\lambda$. Precisely near $\Delta \sim -1$, the expression for $Z'(0)$ behaves as $\exp[-\pi^2/2(\Delta^2-1)^{1/2}]$. Equation (5) is extremely similar to $Z'(0) = -V$. Its right-hand side should perhaps also have an Eulerian product expansion, because the appearance of the π^2 constant is rather impressive. On the left-hand side V would be replaced by the 'staggering' factor $(x^2-1)(x^{-2}-1)$!

The occurrence of Eulerian products corresponding to the expressions of several order parameters of the six-vertex model (spontaneous polarisation (Baxter 1973), magnetisation discontinuity (Jaekel *et al* 1985) as well as in two-dimensional soluble models (Baxter *et al* 1975, Baxter 1982a, b, Andrews *et al* 1984) seems to reinforce this point of view. In general one may expect that the spin-wave-vertex operators $O_{N,N}$ of Kohmoto *et al* might lead to similar Eulerian product forms. Thus, this class of bifurcating critical lines corresponds to a very different mechanism and cannot be considered as integrability lines (and hence algebraic).

3. Comments, speculations and conclusions

The study of the automorphy group associated to the (anisotropic) SAT model has exhibited the algebraic varieties stable by that infinite discrete group. One gets three such algebraic varieties: one is the already known self-dual line, another is a line which is the T transform of the previous one and the third one is an algebraic curve of order four in ω and ω''. All of these three algebraic varieties correspond to conditions for the exact solubility of the model. Only the self-dual line lies in the physical domain. On the other hand it has been emphasised that the other critical lines (manifolds) correspond to a completely different mechanism: a transition in the optimum choice for the corresponding order parameter which becomes different from zero and leads to a non-algebraic manifold. The relevance of the Eulerian product for the expression of these manifolds is emphasised. This work suggests several extensions and raises new questions.

(*a*) The non-symmetric Ashkin–Teller model is a straightforward extension of the previous analysis (one has two algebraic invariants to consider instead of the only one Δ and therefore one is led to an elliptic uniformisation instead of a rational one).

(*b*) The generalisation of the SAT model to $\mathbb{Z}_N \oplus \mathbb{Z}_M$ spin models (as proposed by Zamolodchikov and Monastyrskii 1979) may bring about new features: it does not seem possible to exhibit any algebraic invariant Δ to foliate the space of parameters.

From a more general point of view the set of critical points (or zeros of the partition function for complex values of the parameters) seem to lie on some remarkable manifolds (or even algebraic varieties) invariant under some $\mathbb{C}^n \to \mathbb{C}^n$ mapping generating a part of the automorphy group (when an inversion relation exists for the model) for many model on lattices (in opposition with some Julia set-like distribution of these points). In that respect, it is remarkable for a model like the three-dimensional Ising model for which an inversion relation exists that the set of zeros of the partition function studied numerically (Pearson 1982) seems to lie on a smooth curve: is that curve an algebraic one?

References

Alcaraz F and Köberle R 1981 *J. Phys. A: Math. Gen.* **14** 1169
Andrews G E, Baxter R J and Forrester R J 1984 *J. Stat. Phys.* **35** 193
Ashkin J and Teller E 1943 *Phys. Rev.* **64** 1788
Ashley S E 1978 *J. Phys. A: Math. Gen.* **11** 2015
Baltar V I, Carneiro G M, Pol M E and Zagury N 1984 *J. Phys. A: Math. Gen.* **17** 2119
Baxter R J 1973 *J. Phys. C: Solid State Phys.* **6** L445
—— 1982a *J. Stat. Phys.* **28** 1
—— 1982b *Exactly Solved Model in Statistical Mechanics* (New York: Academic)
Baxter R J, Sykes M F and Watts M G 1975 *J. Phys. A: Math. Gen.* **8** 245
Baxter R J, Temperley H N V and Ashley S E 1976 *Proc. R. Soc.* A **358** 8535
Belavin A B, Polyakov A M and Zamolodchikov A B 1984 *J. Stat. Phys.* **34** 763
Creutz M and Okawa M 1983 *Nucl. Phys.* B **220** 149
Ditzian R V, Banavar J R, Crest G S and Kadonoff L P 1980 *Phys. Rev.* B **22** 2542
Fan C 1972a *Phys. Lett.* **39A** 136
—— 1972b *Phys. Rev.* B **6** 902
Igloi F and Solyom J 1984 *J. Phys. A: Math. Gen.* **17** 1531
Jaekel M T and Maillard J M 1982 *J. Phys. A: Math. Gen.* **15** 2241
Jaekel M T, Maillard J M and Rammal R 1985 *Phys. Rev. Lett.* **54** 209
Kadanoff L 1980 *Phys. Rev.* B **22** 1405
Kohmoto M, den Nijs M and Kadanoff L 1981 *Phys. Rev.* B **24** 5829
Lieb E and Wu F Y 1972 *Phase Transitions and Critical Phenomena* vol 1, ed C Domb and M S Green (New York: Academic) pp 417, 427
Pearson R B 1982 *Phys. Rev.* B **26** 6285
Rujan P, Williams G O, Frisch H L and Forgacs G 1981 *Phys. Rev.* B **23** 1362
Schultz C L 1981 *Phys. Rev. Lett.* **46** 629
Stroganov Y G 1979 *Phys. Lett.* **74A** 116
Temperley H N V and Lieb E 1971 *Proc. R. Soc.* **322** 251
Truong T 1984 *J. Phys. A: Math. Gen.* **17** L473
—— 1985 *Seminar in Non-Linear Equations in Field Theory* ed N Sanchez and H de Vega (Berlin: Springer)
Wegner F J 1972 *J. Phys. C: Solid State Phys.* **5** L131
Wu F Y 1977 *J. Math. Phys.* **18** 611
Wu F Y and Lin K Y 1975 *Phys. Rev.* B **12** 419
Zamolodchikov A B and Monastyrskii M J 1979 *Sov. Phys.-JETP* **50** 117